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Squeezing Dynamics for One-Dimensional Infinite Square Well with a Mobile Wall

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Abstract We show that the dynamics for a particle confined in one-dimensional infinite square well with a mobile boundary can be converted to the case as if the boundary is time-independent at the expense of an appropriate time-dependent Hamiltonian. The Hamiltonian is deduced by the technique of integration within an ordered product of operators.

Keywords Infinite square well · Mobile wall · IWOP technique

1 Introduction

Almost in every quantum mechanics textbook the quantization of a moving particle's energy in a one-dimensional infinite square well is discussed [1, 2]. In this work we consider a particle with mass m confined in the interval $x=0$ to $x=W(t)$ of an infinite square well, i.e., one wall of the well is fixed and the other is mibile with time t. The corresponding Shrödinger equation is

$$i\frac{\partial}{\partial t}|\psi\rangle = -\frac{1}{2m}\frac{\partial^2}{\partial x^2}|\psi\rangle = \frac{P^2}{2m}|\psi\rangle, \tag{1}$$

where we let $\hbar = 1$, and the wave functions vanish at the walls, so they satisfy the boundary condition

$$\psi(0,t) = \psi[x=W(t),t] = 0. \tag{2}$$

From the normalization condition

$$\int_0^{W(t)} \psi^*(x,t)\psi(x,t)dx = 1, \tag{3}$$

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and its invariance with time we also have

$$\frac{\partial}{\partial t}\int_0^{W(t)} \psi^*(x,t)\psi(x,t)dx = 0. \tag{4}$$

This is guaranteed since the left hand side of (4) is equal to

$$\int_0^{W(t)} \frac{\partial}{\partial t}|\psi(x,t)|^2 dx + \frac{\partial W(t)}{\partial t}|\psi(W(t),t)|^2 = 0. \tag{5}$$

In this work we shall show that dynamics for a particle confined in one-dimensional infinite square well with a mobile boundary can be converted to the case as if the boundary is time-independent at the expense of an appropriate time-dependent Hamiltonian. We shall derive this Hamiltonian by virtue of the technique of integration within an ordered product of operators [3, 4].

2 The Dilation Transformation for the Mobile Wall

When we set

$$\bar{x}(t) = x\frac{W(0)}{W(t)} \equiv x\mu(t), \tag{6}$$

(3) can be rewritten as

$$\int_0^{W(0)} \left|\psi\left(\frac{\bar{x}(t)}{\mu(t)}, t\right)\right|^2 d\frac{\bar{x}(t)}{\mu(t)} = 1, \tag{7}$$

and the corresponding momentum operator becomes

$$\bar{P}(t) = P\frac{W(t)}{W(0)} \equiv P/\mu(t) \tag{8}$$

to make up $[\bar{x}(t), \bar{P}(t)] = i$. Comparing (7) with (3) we see that the mobile boundary value $x = W(t)$ becomes fixed value, $\bar{x} = W(0)$, but the wave function $\psi(x,t)$ becomes

$$\psi(\bar{x}(t), t) = \psi(x\mu(t), t) = \langle x\mu(t)|\psi, t\rangle. \tag{9}$$

Supposing that the unitary transformation is

$$UXU^\dagger = \frac{X}{\mu(t)}, \qquad UPU^\dagger = P\mu(t), \tag{10}$$

we apply the $U^\dagger$ transformation to the both sides of (1) and let $U^\dagger|\psi\rangle = |\phi\rangle$, such that

$$\langle x|\phi\rangle = \sqrt{\mu(t)}\langle \mu(t)x|\psi\rangle = \langle x|U^\dagger|\psi\rangle, \tag{11}$$

so

$$\begin{aligned} iU^\dagger\frac{\partial}{\partial t}|\psi\rangle &= U^\dagger\frac{P^2}{2m}|\psi\rangle = \frac{P^2}{2m\mu^2(t)}U^\dagger|\psi\rangle \\ &= i\left[\frac{\partial(U^\dagger|\psi\rangle)}{\partial t} - \frac{\partial U^\dagger}{\partial t}|\psi\rangle\right] = i\frac{\partial}{\partial t}|\phi\rangle - i\frac{\partial U^\dagger}{\partial t}U|\phi\rangle, \end{aligned} \tag{12}$$

and

$$i\frac{\partial}{\partial t}|\phi\rangle = \left[\frac{P^2}{2m\mu^2(t)} + i\frac{\partial U^\dagger}{\partial t}U\right]|\phi\rangle. \tag{13}$$

Comparing with the Shrödinger equation $i\frac{\partial}{\partial t}|\phi\rangle = H(t)|\phi\rangle$ for obtaining the time-dependent Hamiltonian

$$H(t) = \frac{P^2}{2m\mu^2(t)} + i\frac{\partial U^\dagger}{\partial t}U, \tag{14}$$

so we must know $\frac{\partial U^\dagger}{\partial t}$.

3 The Derivation of $\frac{\partial U^\dagger}{\partial t}$ and $H(t)$

Equations (10), (12) are named dilation transformation, the corresponding squeezing operator is [1]

$$U = C\int_{-\infty}^{\infty} dx|\bar{x}(t)\rangle\langle x|, \tag{15}$$

where C is anticipated for U's unitarity. Since $UU^\dagger = 1$, we have

$$\begin{aligned} UU^\dagger &= |C|^2\int_{-\infty}^{\infty} dx|\bar{x}(t)\rangle\langle x|\int_{-\infty}^{\infty} dx'|x'\rangle\langle\bar{x}'(t)| \\ &= |C|^2\int_{-\infty}^{\infty} dx|\bar{x}(t)\rangle\langle\bar{x}(t)| \\ &= |C|^2\int_{-\infty}^{\infty} |\bar{x}(t)\rangle\langle\bar{x}(t)|\frac{d\bar{x}}{\mu(t)} = \frac{|C|^2}{\mu(t)} = 1, \end{aligned} \tag{16}$$

so the squeezing operator is

$$U = \sqrt{\mu(t)}\int_{-\infty}^{\infty} dx|x\mu(t)\rangle\langle x|. \tag{17}$$

Remember that in the Fock space the coordinate eigenvector is

$$|x\rangle = \pi^{-1/4}\exp\left\{-\frac{x^2}{2} + \sqrt{2}xa^\dagger + \frac{a^{\dagger 2}}{2}\right\}|0\rangle, \qquad X|x\rangle = x|x\rangle, \tag{18}$$

where $|0\rangle$ is the vacuum state, $X = (a^\dagger + a)/\sqrt{2}$, and the normally ordered form of $|0\rangle\langle 0|$ is $:\exp\{-a^\dagger a\}:$, so

$$|x\mu(t)\rangle\langle x| = \sqrt{\frac{1}{\pi}}:e^V:, \tag{19}$$

where

$$:e^V: = \,:\exp\left[-\frac{x^2}{2}(1+\mu^2) + \sqrt{2}x(\mu a^\dagger + a) - \frac{1}{2}(a + a^\dagger)^2\right]:. \tag{20}$$

Thus U's time differentiation is

$$\frac{\partial U}{\partial t}=\frac{\dot{\mu}}{2\mu}U+\sqrt{\frac{\mu}{\pi}}\frac{\partial}{\partial t}\int_{-\infty}^{\infty}dx{:}e^{V}{:}, \tag{21}$$

where

$$\frac{\partial}{\partial t}{:}e^{V}{:}=\ {:}(-x^2\mu\dot{\mu}+\sqrt{2}x\dot{\mu}a^{\dagger})e^{V}{:}. \tag{22}$$

On the other hand, from

$${:}\frac{\partial}{\partial a}f(a,a^{\dagger}){:}=[{:}f(a,a^{\dagger}){:},a^{\dagger}], \tag{23}$$

we obtain

$${:}\frac{\partial}{\partial a}e^{V}{:}=\ {:}\big(\sqrt{2}x-a-a^{\dagger}\big)e^{V}{:}, \tag{24}$$

and

$${:}\frac{\partial^2}{\partial a^2}e^{V}{:}=\ {:}\big[\big(\sqrt{2}x-a-a^{\dagger}\big)^2-1\big]e^{V}{:}. \tag{25}$$

Using (23)–(25) we can re-express (22) as

$$\begin{aligned}\frac{\partial}{\partial t}{:}e^{V}{:}&={:}-\frac{\mu\dot{\mu}}{2}\left[\frac{\partial^2}{\partial a^2}+2(a+a^{\dagger})\frac{\partial}{\partial a}+(a+a^{\dagger})^2+1\right]e^{V}{:}\\&\quad+{:}\dot{\mu}a^{\dagger}\left(\frac{\partial}{\partial a}+a+a^{\dagger}\right)e^{V}{:}\\&=-\frac{\mu\dot{\mu}}{2}\big\{\big[[{:}e^{V}{:},a^{\dagger}],a^{\dagger}\big]+2a^{\dagger}[{:}e^{V}{:},a^{\dagger}]+2[{:}e^{V}{:},a^{\dagger}]a+{:}\big[(a+a^{\dagger})^2+1\big]e^{V}{:}\big\}\\&\quad+\dot{\mu}a^{\dagger}\big\{[{:}e^{V}{:},a^{\dagger}]+a^{\dagger}{:}e^{V}{:}+{:}e^{V}{:}a\big\}.\end{aligned} \tag{26}$$

Using

$$\big[[{:}e^{V}{:},a^{\dagger}],a^{\dagger}\big]=\ {:}e^{V}{:}a^{\dagger 2}-2a^{\dagger}{:}e^{V}{:}a^{\dagger}+a^{\dagger 2}{:}e^{V}{:}, \tag{27}$$

and

$${:}\big[(a+a^{\dagger})^2+1\big]e^{V}{:}=a^{\dagger 2}{:}e^{V}{:}+{:}e^{V}{:}a^2+2a^{\dagger}{:}e^{V}{:}a+{:}e^{V}{:}, \tag{28}$$

we obtain

$$\begin{aligned}\frac{\partial}{\partial t}{:}e^{V}{:}&=-\frac{\mu\dot{\mu}}{2}{:}e^{V}{:}(a+a^{\dagger})^2+\dot{\mu}a^{\dagger}{:}e^{V}{:}(a+a^{\dagger})\\&=-\mu\dot{\mu}{:}e^{V}{:}X^2+\sqrt{2}\dot{\mu}a^{\dagger}{:}e^{V}{:}X.\end{aligned} \tag{29}$$

Substituting (26) into (21) we see

$$\begin{aligned}\frac{\partial U}{\partial t}&=\frac{\dot{\mu}}{2\mu}U-\mu\dot{\mu}UX^2+\sqrt{2}\dot{\mu}a^{\dagger}UX\\&=\left(\frac{\dot{\mu}}{2\mu}-\frac{\dot{\mu}}{\mu}X^2+\sqrt{2}\frac{\dot{\mu}}{\mu}a^{\dagger}X\right)U\end{aligned}$$

$$= \frac{\dot{\mu}}{2\mu}(a^{\dagger 2} - a^2)U. \tag{30}$$

Thus (14) becomes (noticing $P = (a - a^\dagger)/(i\sqrt{2})$)

$$H(t) = \frac{P^2}{2m\mu^2(t)} + i\frac{\dot{\mu}}{2\mu}U^\dagger(a^2 - a^{\dagger 2})U = \frac{P^2}{2m\mu^2(t)} + i\frac{\dot{\mu}}{2\mu}(XP + PX), \tag{31}$$

with the normalization

$$\int_0^{W(0)} |\phi(\bar{x})|^2 d\bar{x} = 1. \tag{32}$$

In summary, we have used the IWOP technique to study the squeezing dynamics for one-dimensional infinite square well with a mobile wall.

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